General Relativity | Week 8

Schwarzschild (exterior) spacetime:

$(\mathcal{M}, g_M)$,

$$g_M = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 \underbrace{g_{S^2}}_{d\theta^2 + \sin^2\theta\, d\phi^2}$$

$$\mathcal{M} = \mathbb{R}_t \times (2M, +\infty)_r \times S^2_{\theta\phi} \quad \text{if } M > 0$$

- Spherically symmetric: The action of $SO(3)$ on the S^2 factor is by isometries
- Static: ∂_t is Killing and hypersurface orthogonal
- Asymptotically flat: $g_M = \eta + O(r^{-1})$ as $r \to +\infty$.

(Has ~~conformal boundary~~ $\mathcal{I}^{\pm}$ like Minkowski spacetime)

Killing algebra: 4 dimensional (translations in t & rotations)

What happens at $r = 2M$? Is it a singularity?

In the case of the simpler metric

$g = -t^2 dt^2 + dx^2$: $t = 0$ looks like a singularity, but if we set $\tilde{t} = \frac{1}{2}t^2$, then $g = -d\tilde{t}^2 + dx^2$

Non-smooth change of variables

Similarly, in the usual (θ, ϕ) coordinate system on S^2:
The round metric looks singular, but this is only a coordinate singularity.

An intrinsic approach to check if $r = 2M$ is a "true" singularity or a "coordinate" one: Look for coordinate systems with some geometric significance, such as the ones constructed using null geodesics (still I might encounter problems, e.g. caustics).

(In the Riemannian setting: the question of smooth extendability can be easily addressed, using the Riemannian distance function)

Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a null geodesic in $(\mathcal{M}, g_M)$

As we will see in the exercises: $(\theta(s), \phi(s))$ moves along a great circle (you can deduce this by noting that $\langle \dot\gamma, \Omega \rangle = \text{const}$ for any spherical rotation v.f. Ω).

So: w.l.o.g: $\theta = \frac{\pi}{2} = \text{const.}$

Then, conserved quantities:
- $\langle \dot\gamma, \dot\gamma \rangle = 0$
- $\langle \dot\gamma, \partial_t \rangle = -E$ (energy)
- $\langle \dot\gamma, \partial_\phi \rangle = L$ (angular momentum)

Enough conserved quantities to reduce to 1st order system:

$E = (1 - \frac{2M}{r}) \dot t$, $L = r^2 \dot\phi$ so

$$0 = \langle \dot\gamma, \dot\gamma \rangle = -\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot r^2}{1-\frac{2M}{r}} + \frac{L^2}{r^2} \Rightarrow \boxed{\dot r^2 = E^2 - \frac{1-\frac{2M}{r}}{r^2} L^2}$$

If $L = 0$: (radial null geodesic) and $\dot r(0) < 0$, $E > 0$ (future directed, ingoing)

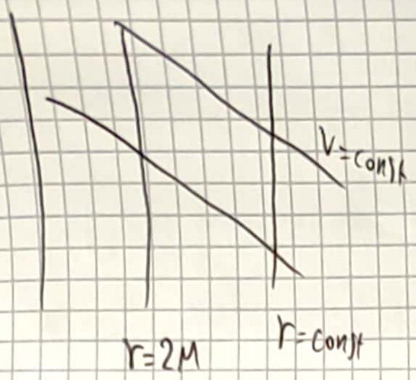
Then $\dot r = \text{const} = -E$ so we reach $r = 2M$ in finite affine time!

If $r^* := r + 2M \log(r - 2M)$ $\left(\text{So } \frac{dr_*}{dr} = \frac{1}{1-\frac{2M}{r}} \right)$

and $v = t + r^*$

Then along ingoing radial null geodesic: $\frac{dv}{ds} = \dot t + \frac{dr^*}{dr} \dot r = 0$

($v = \text{const}$ are null hypersurfaces)



In (v, r, θ, ϕ) coordinates

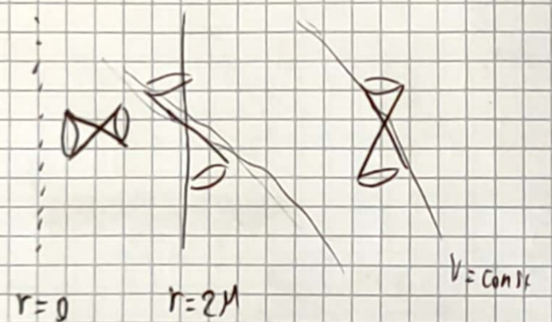
(Eddington-Finkelstein)

$$g = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Smooth across $r=2M$! (∂_r: null, tangent to $\{v=\text{const}\}$)

- Note: $\partial/\partial r$ depends on which are the rest of the coordinates.

In this system $\frac{\partial}{\partial v}$ is the static Killing vector field.



We can extend to $\tilde{M} = \mathbb{R}_v \times (0,+\infty)_r \times S^2_{\theta,\phi}$

from $M = \mathbb{R}_v \times (2M,+\infty)_r \times S^2_{\theta,\phi}$

Note:
- If $r_0 < 2M$: The $\{r=r_0\}$ hypersurfaces are spacelike
- If we start from $r=2M$: No future directed causal curve can escape! $\Longrightarrow$ Black hole!
- Ingoing radial null geodesics "hit" $r=0$ in finite affine time.

This is a true singularity! The Kretschmann scalar

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{C}{r^6} \xrightarrow{r\to 0} +\infty$$

Coordinate independent!

So we cannot extend g as a C^2 metric beyond $r=0$

(Sbierski: No C^0 extension)

What is the "maximal" extension we can construct?

Kruskal-Szekeres:

Double-null coordinates:

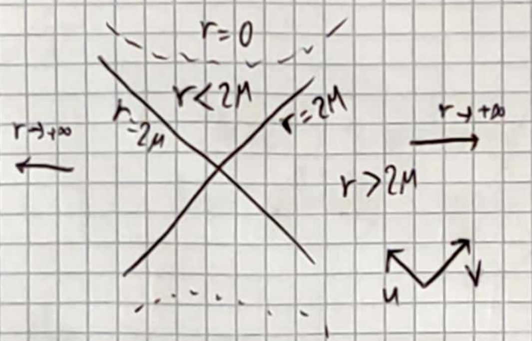
(u,v): $\quad \frac{v}{u} = -e^{-\frac{t}{2M}}, \quad v \cdot u = \left(1-\frac{r}{2M}\right) e^{r/2M} \quad \left(= -\frac{1}{2M} e^{r_*/2M} \right)$

$$\left(\text{as } r \to +\infty: \quad \begin{aligned} \log(-u) &= \frac{t+r_*}{4M} - \frac{1}{2}\log(2M) \\ \log(v) &= \frac{-t+r_*}{4M} - \frac{1}{2}\log(2M) \end{aligned} \right)$$

then $g_M = -32\,\frac{M^3}{r} \exp\left(-\frac{r}{2M}\right) du\,dv + r^2 g_{S^2}$

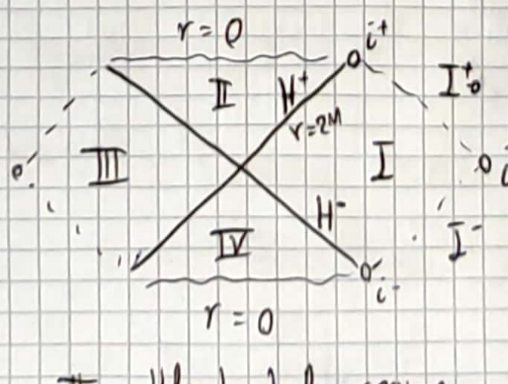
$$\boxed{u \cdot v < 1}$$

~~Penrose diagram~~:



Setting $\tilde{u} = \tilde{u}(u)$, $\tilde{v} = \tilde{v}(v)$ to compactify:

Penrose diagram:



I: Original region, II: Black hole region

I ≅ III, IV: White hole.

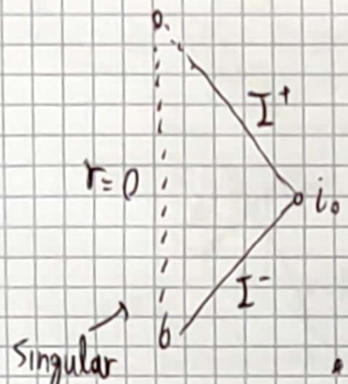
- Trapped null geodesics: go to i^+.
- Note: The image of a causal curve on the Penrose diagram is causal with respect to the background 1+1 metric
- Draw the $r=$const, $t=$const hypersurfaces

The maximally extended Schwarzschild:

- Globally hyperbolic, Cauchy hypersurface $\simeq \mathbb{R} \times S^2$
- Asymptotically flat with 2 as. flat ends.

Exercise: The region I & II is also globally hyperbolic. Draw a Cauchy hypersurface. Is it complete as a Riemannian manifold?

$M < 0$: (Unphysical)



- "Naked" singularity: Visible from $\mathcal{I}^+$
- Not globally hyperbolic

M has the meaning of mass:

- For a massive (timelike) observer: $\langle \dot\gamma, \dot\gamma \rangle = -\mu$

  Geodesic equation: $\dot r^2 = E^2 - \frac{1-\frac{2M}{r}}{r^2} L^2 - (1 - \frac{2M}{r})\mu$

  For $r \gg 2M$: Trajectory similar to Newtonian orbit for gravitational potential $\frac{M}{r}$.

Birkhoff's theorem

Every spherically symmetric (M^{3+1}, g) solution to the vacuum Einstein equations $Ric = 0$ is locally isometric to (M_{Sch}, g_M) for some $M \in \mathbb{R}$.

- No degrees of freedom in spherical symmetry

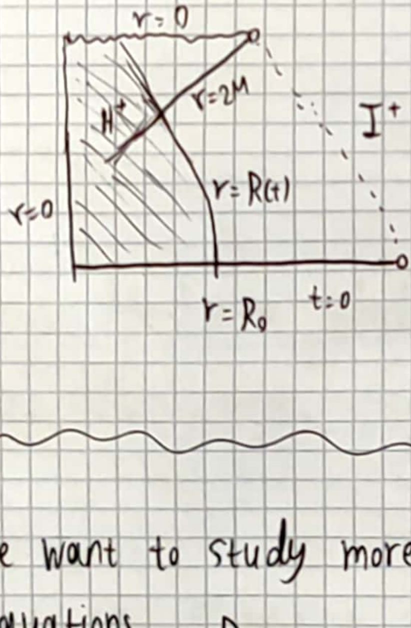
Schwarzschild black hole: "eternal" - 2-ended.

However, black holes can form dynamically:

Oppenheimer - Snyder (1939)

For a sph. symmetric, homogeneous ball of dust ($\leftarrow$ pressureless fluid)

$(T_{\mu\nu} = \rho \cdot u_\mu u_\nu)$



By Birkhoff's theorem, the vacuum exterior has to be Schwarzschild!

We want to study more general solutions of the vacuum Einstein equations $Ric = 0$

Written in coordinates:

$$\underbrace{-\tfrac{1}{2}\Box_g g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)}}_{\text{principal symbol!}} + g\cdot\partial g\cdot\partial g = 0 \qquad \to \Gamma_\nu = g_{\mu\nu}\, g^{\alpha\beta}\,\Gamma^\mu_{\alpha\beta} \sim \partial g$$

$$\Box_g f := \frac{1}{\sqrt{-\det g}}\,\partial_\alpha\left(\sqrt{-\det g}\; g^{\alpha\beta}\,\partial_\beta f\right) = \operatorname{div}(df)$$

: Laplace - Beltrami (wave) operator

Note $\Box_g$ is of hyperbolic signature

When $g = \eta$: $\Box_\eta = -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2$

(classical wave operator)

① is not of some well-known type (eg elliptic, hyperbolic etc)

(if it was, we wouldn't have the freedom of changing coordinates).

In wave coordinates: $\Box_g x^\mu = 0 \Leftrightarrow g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0 \Leftrightarrow \Gamma_\nu = 0$

$$① \rightsquigarrow \begin{cases} \Box_g g = Q(g, \partial g) \\ \Gamma_\nu = 0 \end{cases}$$

$\leftarrow$: Quasilinear wave eqn. (system)

$\leftarrow$ Overdetermined (this will restrict our freedom to choose initial data.

Reissner-Nordström black hole:

Spherically symmetric / static:

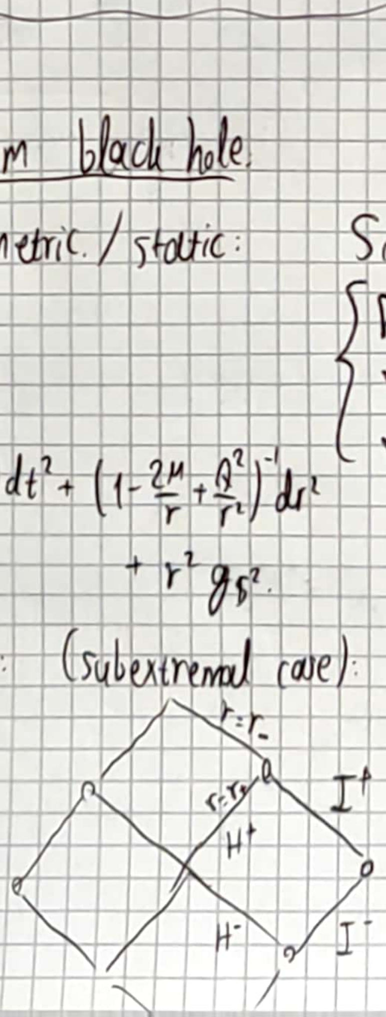
Solution of Einstein-Maxwell system:

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}[F] \\ \nabla^\mu F_{\mu\nu} = 0 \\ \nabla_{[\alpha} F_{\beta\gamma]} = 0 \end{cases}, \quad T_{\mu\nu}[F] = F_\mu{}^\lambda F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}$$

Exterior metric:

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 g_{S^2}.$$

When $0 < |Q| < M$ (subextremal case): Maximal globally hyperbolic development:



$r_\pm$: roots of $r^2 - 2Mr + Q^2 = 0$

At $r = r_-$: The metric smoothly extendible (Cauchy horizon)

But inextendible as globally hyperbolic spct.